Assessing the Error in a Finite Element Solution

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Abstract—**In this paper a method of error assessment for the finite element method is discussed. This idea is used to optimize grid refinement schemes both for singularities and also for handling unbounded regions. It is shown how those elements, or groups of elements, that make large contributions to the error term can be identified so that local grid refinements can be placed in the most advantageous regions.**

I. Introduction

THE FINITE element method for solving field problems has become a widely used procedure (see, for example [1], [2]). In this paper some ideas concerning error assessment are presented with reference to a particular problem in electrical field theory.

The problem is illustrated in Fig. 1 and is a parallel plate capacitor in free space (permittivity ε_0) with a dielectric medium of permittivity ε_1 between the plates which are of width $2a$ and separation $2b$. The electric field u of this problem satisfies the differential equation

$$\nabla \cdot \varepsilon \nabla u = 0, \quad \text{in } \Omega \tag{1}$$

where

i) Ω is the region $-\infty < x < \infty$, $-\infty < y < \infty$,
ii) $\varepsilon(x, y) = \varepsilon_1$ in the dielectric,
$= \varepsilon_0$ elsewhere

subject to the boundary conditions

i) $u = +V$ on plate 1
ii) $u = -V$ on plate 2
iii) $u \to 0$ as x or $y \to \pm\infty$.

The calculation of u and of the capacitance C

$$C = \frac{1}{4V^2} \int \varepsilon |\nabla u|^2 \, d\Omega \tag{2}$$

are important for this problem.

II. Error Assessment

The method of error assessment adopted here is related to the use of dual or complementary variational principles and Synge's hypercircle method, [3]–[5]. To illustrate the method we consider the differential equation (1), for which Ω is a finite two-dimensional region with boundary $\partial\Omega$, subject to the boundary conditions

i) $u = f$ on $\partial\Omega_1$ (3)

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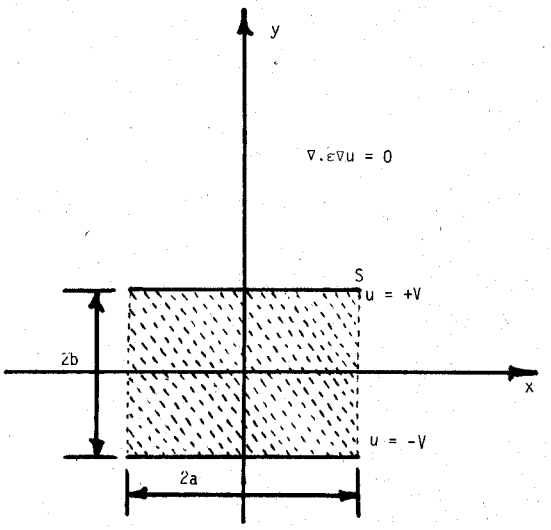

Fig. 1. A parallel plate capacitor.

and

ii) $\partial u / \partial n = 0$ on $\partial\Omega_2$ (4)

where $\partial\Omega_1$ and $\partial\Omega_2$ are parts of the boundary of Ω.

The solution of this differential equation gives the minimum to the functional

$$J''(\phi'') = \int_\Omega \varepsilon |\nabla \phi''|^2 \, d\Omega \tag{5}$$

over all functions belonging to $D''(\Omega)$ where $D''(\Omega)$ is the set of functions that

i) are continuous in Ω and have square integrable first derivatives in Ω,
ii)satisfy the boundary condition (3).

The dual or complementary approach to this problem is to find the minimum v of the functional

$$J'(\phi') = \int_\Omega \frac{1}{\varepsilon} |\nabla \phi'|^2 \, d\Omega - 2 \int_{\partial\Omega_1} f \frac{\partial \phi}{\partial s} d(\partial\Omega) \tag{6}$$

over all functions belonging to $D'(\Omega)$ where $D'(\Omega)$ is the set of functions that

i) are continuous in Ω and have square integrable first derivatives in Ω,
ii)are constant along $\partial\Omega_2$, (different constants along distinct portions of $\partial\Omega_2$).

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The solution u of the differential equation is related to v by

$$\frac{\partial u}{\partial x} = \frac{1}{\varepsilon}\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{1}{\varepsilon}\frac{\partial v}{\partial x}. \tag{7}$$

To obtain our error estimate from these two solutions we use the space $H(\Omega)$ of two-dimensional vector-valued functions in Ω with inner product

$$(\mathbf{P},\mathbf{Q})_H = \int_\Omega \varepsilon \mathbf{P}\cdot\mathbf{Q}\, d\Omega \tag{8}$$

which is properly defined when

$$0 < \varepsilon_{\min} \leq \varepsilon(x,y) \leq \varepsilon_{\max} < \infty. \tag{9}$$

The gradient of u is denoted by $\mathbf{S}$ in $H(\Omega)$, the gradient of the typical ϕ'' of $D''(\Omega)$ is denoted by $\mathbf{S}''$ in $H(\Omega)$, and $\mathbf{S}'$ in $H(\Omega)$ denotes

$$\mathbf{S}' = \frac{1}{\varepsilon}\begin{bmatrix} \partial\phi'/\partial y \\ \partial\phi'/\partial x \end{bmatrix} \tag{10}$$

for the typical ϕ' of $D'(\Omega)$, then for all $\mathbf{S}'$ and $\mathbf{S}''$ we have

$$\text{i) } (\mathbf{S}-\mathbf{S}',\mathbf{S}-\mathbf{S}')_H \leq (\mathbf{S}'-\mathbf{S}'',\mathbf{S}'-\mathbf{S}'')_H \equiv E \tag{11}$$

$$\text{ii) } (\mathbf{S}-\mathbf{S}'',\mathbf{S}-\mathbf{S}'')_H \leq E \tag{12}$$

$$\text{iii) } (\mathbf{S}'',\mathbf{S}'')_H - E \leq (\mathbf{S},\mathbf{S})_H \leq (\mathbf{S}'',\mathbf{S}'')_H. \tag{13}$$

Thus E is an upper bound of the error in $H(\Omega)$ between both $\mathbf{S}$ and $\mathbf{S}'$, and $\mathbf{S}$ and $\mathbf{S}''$.

As defined, this estimate E of the error is seldom of great value although if E is zero then both $\mathbf{S}'$ and $\mathbf{S}''$ must represent the solution $\mathbf{S}$ and if E is very small compared with $(\mathbf{S}',\mathbf{S}')_H$ and $(\mathbf{S}'',\mathbf{S}'')_H$ then both $\mathbf{S}'$ and $\mathbf{S}''$ must be good approximations. However, E is an integral over the region Ω and, when approximate minima of the functionals (5) and (6) are calculated by the finite element method on the same grid of elements, the element-by-element contributions to E can be calculated and hence element-by-element assessment of accuracy. In this way, those elements making major contributions to the error can be identified and singled out for grid refinement.

These ideas can be extended to problems in which Ω is unbounded. If we take the boundary condition at infinity to be

$$u \to 0 \text{ as } (x,y) \to \infty \tag{14}$$

then we have to add the extra condition to D'' that

$$\phi'' \to 0 \text{ as } (x,y) \to \infty \tag{15}$$

and an extra condition to D; namely

$$\phi' \to (\text{constant}) \text{ as } (x,y) \to \infty. \tag{16}$$

These ideas can be further extended to problems with rotational symmetry and general three-dimensional problems. More generally, the dual principles are nonlinear and the bounds (11) and (12) are not valid. Nevertheless, the calculation of element-by-element contributions to E, as defined by (11) could be used to find those elements in which the representation is likely to be poor.

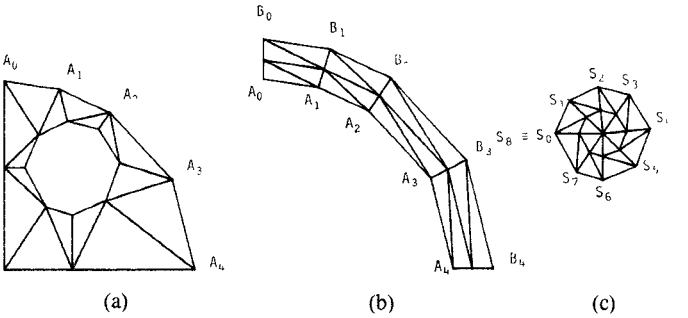

Fig. 2. Grid 1 with refinement at the singularity and 2 "at infinity." (a) The blending section. (b) Refinement for the infinite region. (c) Refinement for the singularity.

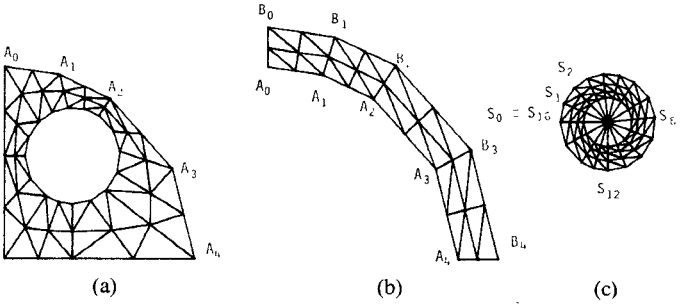

Fig. 3 Grid 2 with 3 refinements at the singularity and 2 "at infinity." (a) The blending section. (b) Refinement for the infinite region. (c) Refinement for the singularity.

III. Finite Element Grids for the Parallel Plate Capacitor

A brief description of a parallel plate capacitor is given in Section I and by exploiting symmetry it can be reduced to solving the problem in the positive quadrant with the extra boundary conditions

$$\text{i) } u = 0 \text{ on the } X\text{-axis} \tag{17}$$

$$\text{ii) } \partial u/\partial n = 0 \text{ on the } Y\text{-axis}. \tag{18}$$

The automatically generated grid of triangles used to solve this problem are made up in three sections

i) a systematic refinement scheme near the singularity
ii) a systematic refinement scheme to cope with the infinite region
iii) a blending triangulation in between.

The various sections of the typical grids are illustrated in Figs. 2 and 3, where the nodes $\{A_i\}_{i=0}^4$ are a distance R_a from (0,0) and the nodes $\{B_i\}_{i=0}^4$ are a distance R_b from (0,0) with

$$R_b = R_a \times k_{n_a} \tag{19}$$

where

i) k_a is some parameter greater than 1
ii) n_a is the number of refinements for the infinite region

and these nodes form the basis for the systematic refinement scheme for the infinite region. Similarly, the nodes (S_i) are a distance R_s from the singularity at S and nodes

TABLE I
OPTIMUM VALUE OF k_s

n_s	Best k_s for $\varepsilon_1 = \varepsilon_0$	Best k_s for $\varepsilon_1 = 10\varepsilon_0$
2	0.36	0.42
4	0.48	0.54
6	0.56	0.61
8	0.61	0.66

are placed concentrically at distances

$$R_s \times k_s^p, \qquad p = 1,2,\cdots,n_s \tag{20}$$

where

i) k_s is some parameter between 0 and 1,
ii) n_s is the number of refinements at S,

and these nodes form the basis for the systematic refinement scheme for the singularity.

It has been shown [6] that given the form of the singularity at S, (which will depend on the dielectric medium between the plates) and the value of n_s, the value of k_s that minimizes the error is independent of a and b, of the number of nodes on the arcs of radii $R_s k_s^p$ and of R_s. These values are given in Table I.

IV. A Solution Obtained by Ignoring the Unbounded Region

Approximate solutions are obtained by minimizing the functionals (5) and (6), respectively, in the sets $D'(\Omega_\infty)$ and $D''(\Omega_\infty)$ where Ω_∞ is the region $x \geq 0$, $y \geq 0$; the integrals in (5) and (6) are evaluated over Ω_∞. In practice the simplest way to tackle this problem is to assume that the solution is zero outside some large radius, which is the approach adopted below. At the end of this section we will use the ideas discussed above to get an assessment of the error this assumption produces. Here we will consider only the case when $a = b = 1$, $V = 1$, and $\varepsilon_1 = \varepsilon_0$, i.e., no dielectric medium between the plates. The solution will be assumed to be zero outside the polygon $B_0B_1B_2B_3B_4$, for which R_b is taken to be 3. (Note, this is not a good choice if a useful solution is required by this approach.) We will denote by Ω_p the subregion of Ω_∞ inside this polygon.

Approximate solutions are obtained by minimizing the functionals (5) and (6) with respect to generalized coordinates of trial functions in the sets $D''(\Omega_p)$ and $D'(\Omega_p)$ using linear triangular elements on the grids shown in Fig. 4. Each grid has a title of the form

$$2.3(0.51)\cdot 4(1.26)$$

which represents grid 2 with $n_s = 3$, $k_s = 0.51$, $n_a = 4$, $k_a = 1.26$, and for all the grids used here $R_s = 0.8$, $R_a = 2.5$. The shaded elements in Fig. 4 are those elements that are making more than twice the average contribution to the error E.

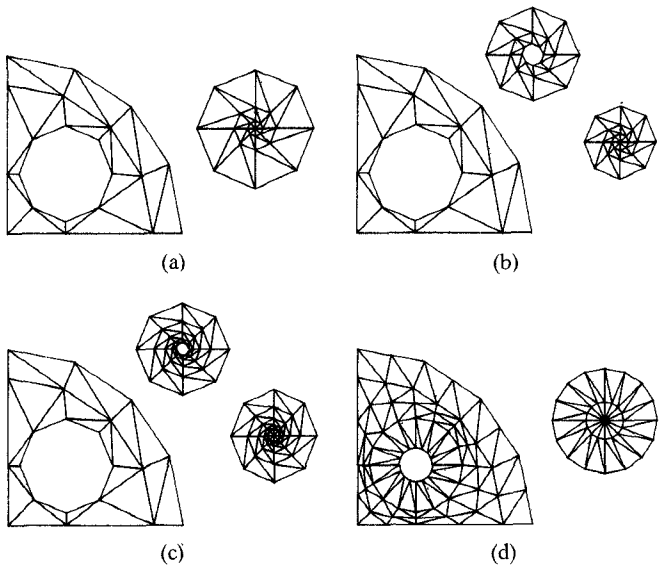

Fig. 4. Typical grids for the restricted problem. (a) Grid 1.2(0.36).1(1.2). (b) Grid 1.4(0.48).1(1.2). (c) Grid 1.8(0.61).1(1.2). (d) Grid 2.2(0.36).1(1.2).

TABLE II

Grid	No. of gen. words	Upper bound on S^2	Lower bound on S^2	Error for restricted problem	Exact lower bound for full problem	True error for full problem
1.2.1	42	23.49	20.76	2.729	15.49	8.005
1.4.1	60	23.16	21.06	2.096	15.66	7.506
1.8.1	96	22.97	21.24	1.741	15.75	7.226
2.2.1	95	22.99	21.30	1.685	15.78	7.206
Spec.	68	22.96	21.39	1.573	15.83	7.127

Columns (3) and (4) are calculated using (13)

Column (5) is calculated using (11).

For the grid 1.2.1, several large elements are making significant contributions to the error but the ratio

$$(\text{element error})/(\text{area of element}) \equiv (\text{element error density}) \tag{21}$$

for these elements is not grossly out of line with the ratio

$$(\text{total error})/(\text{total area}) = (\text{total error density}). \tag{22}$$

However, for the three elements at the singularity the difference in these two ratios is a factor of approximately 50 which indicates that further refinement at the singularity would be beneficial. Thus grid 1.4.1 is used. By looking at column (5), Table II, we can see that the total error has been reduced by about 22 percent and the number of nodes increased by 18.

To improve the solution on grid 1.4.1 without any information about the error we might put in more refinements at the singularity, i.e., use grid 1.8.1, or for a more refined grid to reduce the size of the large elements, i.e., use grid 2.2.1. On both of these grids, the number of nodes has been increased by about 60 percent and the error reduced

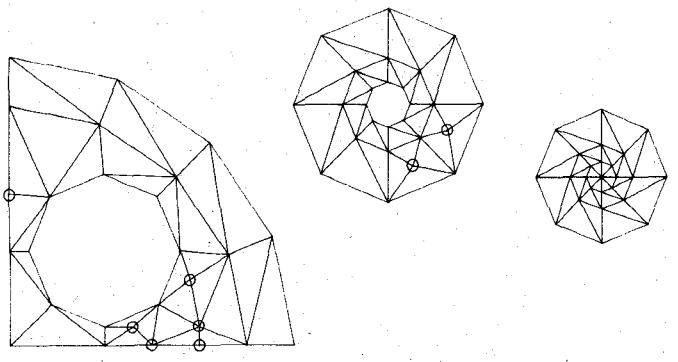

Fig. 5. A special grid. *Note*: a circle around a node indicates that it is an extra node added to the grid 1.4.1, that is an extra eight nodes.

by about 15 percent. But by looking at the elements giving a major contribution to error, that is the shaded elements in Fig. 4(b), we can choose a grid that increases the number of nodes by only 13 percent but decreases the error by 25 percent. The special grid used is illustrated in Fig. 5 and with this grid we get a smaller error than grids 2.2.1 and 1.8.1 but we have added fewer extra nodes. The purpose of this simple example is to illustrate the benefits gained from an element-by-element error estimate. Using this estimate we can choose a grid which gives a good improvement in the solution (i.e., reducing the error) at a relatively small cost (i.e., only slightly increasing the number of nodes).

The function in $D''(\Omega_p)$ can be extended to functions in $D''(\Omega_\infty)$ by defining them to be zero outside Ω_p, thus $D''(\Omega_p)$ is a subset of $D''(\Omega_\infty)$ and if $\mathbf{S}''\in D''(\Omega_p)\subset D''(\Omega_\infty)$, then $(\mathbf{S}'',\mathbf{S}'')_{H(\Omega_p)}=(\mathbf{S}'',\mathbf{S}'')_{H(\Omega_\infty)}$. However, no simple relationship holds between functions in $D'(\Omega_p)$ and $D'(\Omega_\infty)$ because functions in $D'(\Omega_p)$ are unrestricted along the path $B_0B_1B_2B_3B_4$. This difficulty can be overcome by taking a subset $D_0'(\Omega_p)$ of $D'(\Omega_p)$ in which functions are zero along this path. Then, as above, extending these functions to be zero outside Ω_p, $D'_0(\Omega_p)$ is a subset of $D'(\Omega_\infty)$ and if $\mathbf{S}'\in D'_0(\Omega_p)CD'(\Omega_\infty)$ then $(\mathbf{S}',\mathbf{S}')_{H(\Omega_p)}=(\mathbf{S}',\mathbf{S}')_{H(\Omega_\infty)}$. Using trial functions in $D_0'(\Omega_p)$ instead of $D'(\Omega_p)$ we obtain the results in columns (6) and (7) of Table II. The difference between columns (5) and (7) on each grid gives an estimate of the error introduced by restricting the problem to be zero outside R_b. For example, on grid 1.4.1 some 70 percent of the error is produced by the restriction suggesting that to get a better solution more effort should be put into the infinite region rather than at the singularity.

V. Treatment of the Unbounded Region

In the last section we saw that the error introduced by assuming the solution was zero outside some curve which was relatively large. This could be reduced by taking a curve farther away from the origin, but we can do better than arbitrarily choosing some curve. At the singularity, we found an optimum refinement, being able to optimize k_s which was dependent on n_s. Similarly using the sets $D_0'(\Omega_p)$ and $D''(\Omega_p)$ we can optimize the parameter k_a which is dependent on n_a getting the results of Table III. Using these values we get the results of Table IV for the problem with $a=b$ and $\varepsilon_1=\varepsilon_0$. On grid 1.2.2 with 47 nodes, approximately 40 percent of the error is coming from each of the two refinements, suggesting that further refinements would be beneficial. With the grid 1.8.8 the error is spread relatively evenly through the three portions of the grid; further refinement at the singularity and for the unbounded region would reduce the contributions from these two regions but would not greatly reduce the error since the contribution from the blending triangulation would become significant. Out of interest, the results for an infinite triangulation (1.∞.∞) (with $k_a=0.1$, $k_s=0.1$, $R_a=2.5$, $R_s=0.8$), see [7] which is related to the idea discussed by [8], are given. Thus for only 32 nodes we get remarkably good answers. The results for grids 2.4.4 and 2.8.8 are given, although approximately equal contributions to the error arise from the two refinements, these still make major contributions to the total error and would suggest that further refinements would be beneficial. Finally, the results with 2.∞.∞ and 4.∞.∞ are also given.

TABLE III
Optimum Values of k_a

n_a	Optimum k_a
2	1.92
4	1.59
6	1.45
8	1.39

TABLE IV

Grid	n	Upper bound on S^2	Lower bound on S^2	Error in refinement near sing.	Error in refinement for infin.	Error in remaining region	Total Error
1.2.2	47	20.573	17.085	1.390	1.404	0.691	3.489
1.4.4	75	19.899	17.600	0.871	0.861	0.568	2.299
1.8.8	131	19.570	17.876	0.603	0.561	0.530	1.694
1.∞.∞	32	19.344	18.076	0.412	0.344	0.512	1.268
2.4.4	156	19.430	18.061	0.579	0.641	0.148	1.369
2.8.8	260	19.106	18.354 18.354	0.299	0.325	0.127	0.751
2.∞.∞	77	18.882	18.567	0.099	0.097	0.120	0.316
4.∞.∞	215	18.780	18.682	0.029	0.038	0.030	0.097

Note:- the upper and lower bounds on S^2 are equivalent to upper and lower bounds on the capacitance C (given by equation (2)).

Some results for the case when $b=2$, $a=1$, and $\varepsilon_1=10\varepsilon_0$ are given in Table V, in which for each grid $R_a=3.5$, $R_s=0.8$. The singularity is not so severe for this problem, this fact is apparent for the results on both grids 1.2.2 and 2.4.4 since both 47 and 59 percent of the respective errors come from the infinite regions while only 15 percent of the error for both grids comes from the singularity. Thus to

TABLE V

Grid	n	Upper bound on S^2	Lower bound on S^2	Error in refinement near sing.	Error in refinement for infin.	Error in remaining region	Total error
1.2.2	47	55.74	52.02	0.541	1.740	1.441	3.722
1.2.4	57	55.24	52.35	0.465	1.137	1.286	2.888
1.∞.∞	32	54.75	52.76	0.183	0.549	1.259	1.991
2.4.4	156	54.31	53.04	0.193	0.754	0.323	1.270
2.4.8	192	54.10	53.22	0.176	0.403	0.296	0.876
2.∞.∞	77	53.90	53.41	0.044	0.146	0.293	0.483
4.∞.∞	215	53.71	53.57	0.013	0.052	0.074	0.139

For the problem with b = , a = 1, $\varepsilon_1 = 10\varepsilon_0$.

improve the solution on these two grids the best strategy is to place more refinements for the unbounded region, i.e., grids 1.2.4. and 2.4.8.

VI. Conclusion

In this paper some ideas on error assessment in the finite element method have been introduced for field problems. It has been shown how these ideas can be used to choose an optimum grid refinement pattern at a singularity and for an unbounded region. It has also been shown how the effect of arbitrarily assuming a zero solution outside some finite region can be assessed. By looking at element by element contributions to the error, those elements or groups of elements making major contributions to the error can be identified and singled out for refinement.

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Short Papers

Analysis of a Microwave FET Oscillator Using an Efficient Computer Model for the Device

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Abstract—**This paper presents a time domain analysis of a microwave 10-GHz FET oscillator, which employs a practical and efficient computer model for the FET. Good agreement is demonstrated between the predicted and measured performance. A sensitivity analysis of the circuit is performed with respect to some of the FET parameters. This is useful information to estimate performance variation in production.**

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I. Introduction

In the last decade the GaAs MESFET has become an important and useful microwave device. Many microwave components can be built using this device—amplifiers, oscillators, switches, mixers, etc. To enable an accurate and efficient design of components using MESFET's it is useful to have a fast and reasonably accurate large signal model for the device. The basic model of Shockley [1] was shown to be invalid for GaAs short channel FET (modern microwave FET's belong to this category).

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